HOMOGENEOUS SURFACES IN $\mathbb{S}^3$

ARMANDO J. MACCORI AND JOSE A. VERDERESI

Abstract. The goal of this paper is to establish the classification of all homogeneous surfaces of 3-sphere by using the moving frame method. We will show that such surfaces are 2-spheres and flat torus.

Contents

Introduction

In this work we established the classification of all homogeneous surfaces of $\mathbb{S}^3$ by using the method of moving frames. We will denote the 3-sphere by $\mathbb{S}^3$, and it is the following subset of $\mathbb{R}^4$:

$$\mathbb{S}^3 = \left\{ x \in \mathbb{R}^4 \mid \langle x, x \rangle = 1 \right\}.$$

We say that a Riemannian surface S is *homogeneous* if the group $\mathrm{Isom}(S)$ of all isometries of S acts transitively over S, i.e., if $x, y \in S$ are two distinct points of S, then there exists an element $g \in \mathrm{Isom}(S)$ such that $y = g \cdot x$. On the other hand, a surface $S \subset \mathbb{S}^3$ is said to be *extrinsic homogeneous* if the group

$$G = \left\{ g \mid g \in \mathrm{Isom}(\mathbb{S}^3) \text{ and } g(S) \subset S \right\}$$

Date: 2nd February 2008.

2000 *Mathematics Subject Classification.* Primary 53B20, 53C30; Secondary 58A05, 58A10.

Key words and phrases. Homogeneous surfaces, 3-sphere, 2-sphere, flat torus, moving frames, classification theorem.

acts transitively over S. Note that, if we compare these two definitions of homogeneity, it's clear that extrinsic homogeneity implies homogeneity.

We obtained a Classification Theorem for immersed homogeneous surfaces in $\mathbb{S}^3$ (see Section 3). There are only two families of homogeneous surfaces in $\mathbb{S}^3$: the first is composed by 2-*spheres*, given by

$$S = \big\{(x^1,x^2,x^3,x^4) \in \mathbb{R}^4 \mid x^4 = k,\ (x^1)^2 + (x^2)^2 + (x^3)^2 = 1 - k^2\big\},$$

where $0 \leq k < 1$. The second family is composed by *flat torus*. Such surfaces are given by

$$S = \big\{(x^1,x^2,x^3,x^4) \in \mathbb{R}^4 \mid (x^1)^2 + (x^2)^2 = a^2,\ (x^3)^2 + (x^4)^2 = b^2\big\},$$

where $a^2 + b^2 = 1$, and $a, b \in \mathbb{R}$.

1. Structure Equations of $\mathbb{S}^3$

Let (e_1, e_2, e_3, e_4) be a moving frame of $\mathbb{R}^4$ adapted to the sphere $\mathbb{S}^3$, i.e., (e_1, e_2, e_3) belongs to $T\mathbb{S}^3$ and $e_4(x) = -x$. It is easy to see that $(\mathrm{d}e_4)_x = -\mathrm{id}$, since we have

$$\mathrm{d}e_4 = -(\theta^1 e_1 + \theta^2 e_2 + \theta^3 e_3). \tag{1.1}$$

The set (e_1, e_2, e_3, e_4) is an orthonormal frame, so it follows that $\mathrm{d}e_i = \omega_i^k e_k$, where ω_i^k are the connection forms of $\mathbb{R}^4$. Let $(\theta^1, \theta^2, \theta^3, \theta^4)$ be the coframe associated to (e_1, e_2, e_3, e_4), i.e., $\theta^i(e_j) = \delta^i_j$, for $i, j = 1, \ldots, 4$.

The *first structural equations* of $\mathbb{R}^4$ are $\mathrm{d}\theta^i + \omega_k^i \wedge \theta^k = 0$, moreover $\mathrm{d}\theta^4 = 0$ over $\mathbb{S}^3$, and hence our set of equations reduces to

$$\begin{aligned}
\mathrm{d}\theta^1 + \omega_2^1 \wedge \theta^2 + \omega_3^1 \wedge \theta^3 &= 0,\\
\mathrm{d}\theta^2 + \omega_1^2 \wedge \theta^1 + \omega_3^2 \wedge \theta^3 &= 0,\\
\mathrm{d}\theta^3 + \omega_1^3 \wedge \theta^1 + \omega_2^3 \wedge \theta^2 &= 0,
\end{aligned} \tag{1.2}$$

and these equations are called the ***first structural equations*** of $\mathbb{S}^3$. Note that $\mathrm{d}\theta^4 = 0$ implies the important additional condition:

$$\omega_1^4 \wedge \theta^1 + \omega_2^4 \wedge \theta^2 + \omega_3^4 \wedge \theta^3 = 0.$$

The *second structural equations* of $\mathbb{R}^4$ are given by $\mathrm{d}\omega_j^i + \omega_k^i \wedge \omega_j^k = 0$. By the condition (1.1), we obtain $\omega_4^1 = -\theta^1$, $\omega_4^2 = -\theta^2$, $\omega_4^3 = -\theta^3$, and hence

$$\begin{aligned}
\mathrm{d}\omega_2^1 + \omega_3^1 \wedge \omega_2^3 &= \theta^1 \wedge \theta^2,\\
\mathrm{d}\omega_3^2 + \omega_1^2 \wedge \omega_3^1 &= \theta^2 \wedge \theta^3,\\
\mathrm{d}\omega_1^3 + \omega_2^3 \wedge \omega_1^2 &= \theta^3 \wedge \theta^1,
\end{aligned} \tag{1.3}$$

and these equations are called the ***second structural equations*** of $\mathbb{S}^3$. Moreover, the differential 2-forms

$$\begin{aligned} \Omega_2^1 &= \theta^1 \wedge \theta^2, \\ \Omega_3^2 &= \theta^2 \wedge \theta^3, \\ \Omega_1^3 &= \theta^3 \wedge \theta^1, \end{aligned}$$

are called *curvature forms* of $\mathbb{S}^3$.

2. Surfaces in $\mathbb{S}^3$

Let $S \subset \mathbb{S}^3$ be a regular, connected, and oriented surface. Let (e_1, e_2, e_3) be an adapted orthonormal frame to S, i.e., $\langle e_i, e_j \rangle = \delta_{ij}$, where (e_1, e_2) belongs to TS and $e_3 \perp TS$. We have also $\mathrm{d}e_i = \omega_i^k e_k$, where ω_i^j are the connection 1-forms of $\mathbb{S}^3$.

Let $(\theta^1, \theta^2, \theta^3)$ be the coframe associated to (e_1, e_2, e_3). We know that $\theta^3 = 0$ on S, because $e_3 \perp TS$, thus $\mathrm{d}\theta^3 = 0$ on S, and hence the equations in (1.2) reduce to

$$\begin{aligned} &\mathrm{d}\theta^1 + \omega_2^1 \wedge \theta^2 = 0, \\ &\mathrm{d}\theta^2 + \omega_1^2 \wedge \theta^1 = 0, \\ &\omega_1^3 \wedge \theta^1 + \omega_2^3 \wedge \theta^2 = 0, \end{aligned}$$

and these equations are known as the ***first structural equations*** of S. Finally the equations in (1.3) reduce to

$$\begin{aligned} &\mathrm{d}\omega_2^1 + \omega_3^1 \wedge \omega_2^3 = \theta^1 \wedge \theta^2, \\ &\mathrm{d}\omega_3^2 + \omega_1^2 \wedge \omega_3^1 = 0, \\ &\mathrm{d}\omega_1^3 + \omega_2^3 \wedge \omega_1^2 = 0, \end{aligned}$$

and these are called the ***second structural equations*** of S.

Now write

$$\begin{aligned} \omega_1^3 &= h_{11}\theta^1 + h_{12}\theta^2, \\ \omega_2^3 &= h_{21}\theta^1 + h_{22}\theta^2, \end{aligned}$$

and keeping in mind the fact that

$$\omega_1^3 \wedge \theta^1 + \omega_2^3 \wedge \theta^2 = 0,$$

it follows, by Cartan's Lemma, (see do Carmo [1], page 80) that (h_{ij}) is a symmetric matrix. The second fundamental form of surface S is

$$\Pi = \omega_1^3 \cdot \theta^1 + \omega_2^3 \cdot \theta^2 = h_{11}(\theta^1)^2 + 2h_{12}\theta^1\theta^2 + h_{22}(\theta^2)^2,$$

and it is, of course, a diagonalizable operator. In diagonal form, ω_1^3 and ω_2^3 are written

$$\begin{aligned} \omega_1^3 &= \lambda_1\theta^1, \\ \omega_2^3 &= \lambda_2\theta^2, \end{aligned} \tag{2.1}$$

where λ_1, λ_2 are called the *principal curvatures* of S.

From the second structural equations of S we have

$$\mathrm{d}\omega_2^1 + \omega_3^1 \wedge \omega_2^3 = \theta^1 \wedge \theta^2,$$

which implies

$$\mathrm{d}\omega_2^1 = (1 + \lambda_1\lambda_2)\theta^1 \wedge \theta^2, \tag{2.2}$$

and this is called the *Gauss equation* of S. The function

$$K = 1 + \lambda_1\lambda_2,$$

is the *Gaussian curvature* of S. From another pair of equations we have

$$\begin{aligned} \mathrm{d}\omega_3^2 - \lambda_1\omega_1^2 \wedge \theta^1 &= 0, \\ \mathrm{d}\omega_1^3 - \lambda_2\omega_1^2 \wedge \theta^2 &= 0, \end{aligned} \tag{2.3}$$

called the *Mainardi-Codazzi equations* of S.

Differentiating the equations (2.1), we obtain

$$\begin{aligned} \mathrm{d}\omega_1^3 &= \mathrm{d}\lambda_1 \wedge \theta^1 + \lambda_1\mathrm{d}\theta^1, \\ \mathrm{d}\omega_2^3 &= \mathrm{d}\lambda_2 \wedge \theta^2 + \lambda_2\mathrm{d}\theta^2, \end{aligned}$$

Therefore, by the Mainardi-Codazzi equations (2.3), we conclude that

$$\begin{aligned} \lambda_2\omega_1^2 \wedge \theta^2 &= \mathrm{d}\lambda_1 \wedge \theta^1 + \lambda_1\mathrm{d}\theta^1, \\ -\lambda_1\omega_1^2 \wedge \theta^1 &= \mathrm{d}\lambda_2 \wedge \theta^2 + \lambda_2\mathrm{d}\theta^2, \end{aligned} \tag{2.4}$$

On the other hand, the first structural equations said

$$\begin{aligned} \mathrm{d}\theta^1 &= \omega_1^2 \wedge \theta^2, \\ \mathrm{d}\theta^2 &= -\omega_1^2 \wedge \theta^1, \end{aligned} \tag{2.5}$$

so, from (2.4) and (2.5), results that the Mainardi-Codazzi equations will be written in the form

$$\begin{aligned} \mathrm{d}\lambda_1 \wedge \theta^1 + (\lambda_1 - \lambda_2)\omega_1^2 \wedge \theta^2 &= 0, \\ \mathrm{d}\lambda_2 \wedge \theta^2 + (\lambda_1 - \lambda_2)\omega_1^2 \wedge \theta^1 &= 0. \end{aligned} \tag{2.6}$$

Definition **2.1.** Let S and $\tilde{S}$ be two surfaces of $\mathbb{S}^3$. An *isometry* is a diffeomorphism $f : S \longrightarrow \tilde{S}$ which satisfy $\langle f_*(X), f_*(Y)\rangle = \langle X, Y\rangle$, for all pairs $X, Y \in TS$.

Proposition **2.1.** *Let* $(e_1, \ldots, e_n)$ *be a moving frame of a differentiable manifold* M *and let* $(\theta^1, \ldots, \theta^n)$ *be the coframe associated. Then there exists a unique* 1*-forms* ω^i_j *such that*

$$\mathrm{d}\theta^i = \sum_{j=1}^{n} \theta^j \wedge \omega^i_j, \tag{2.7}$$

with the property $\omega^j_i = -\omega^i_j$.

Proof. Let $\tilde{\omega}^i_j$ be 1-forms satisfying equation (2.7). If ω^i_j also satisfies (2.7), then

$$\sum_{j=1}^{n} \theta^j \wedge (\omega^i_j - \tilde{\omega}^i_j) = 0. \tag{2.8}$$

By Cartan's Lemma, from equation (2.8), follows that

$$\omega^i_j - \tilde{\omega}^i_j = \sum_{k=1}^{n} a^i_{jk}\theta^k, \tag{2.9}$$

where a^i_{jk} are symmetric ($a^i_{jk} = a^i_{kj}$). Since $(\theta^1, \ldots, \theta^n)$ is a base for the set of 1-forms in M, then there exist Γ^i_{jk} such that

$$\tilde{\omega}^i_j = \sum_{k=1}^{n} \Gamma^i_{jk}\theta^k. \tag{2.10}$$

Thus, from equations (2.9) and (2.10), follows that

$$\omega^i_j = \sum_{k=1}^{n} (\Gamma^i_{jk} + a^i_{jk})\theta^k. \tag{2.11}$$

If $\omega^j_i = -\omega^i_j$ is satisfied, then the equation (2.11) assumes the form

$$(\Gamma^i_{jk} + a^i_{jk}) + (\Gamma^j_{ik} + a^j_{ik}) = 0,$$

which is equivalent to

$$a^i_{jk} + a^j_{ik} = -(\Gamma^i_{jk} + \Gamma^j_{ik}). \tag{2.12}$$

Cyclic permuting the indices i, j, k in equation (2.12), we write

$$\begin{aligned} a^i_{jk} + a^j_{ik} &= -(\Gamma^i_{jk} + \Gamma^j_{ik}), \\ a^k_{ij} + a^i_{kj} &= -(\Gamma^k_{ij} + \Gamma^i_{kj}), \\ a^j_{ki} + a^k_{ji} &= -(\Gamma^j_{ki} + \Gamma^k_{ji}). \end{aligned} \tag{2.13}$$

In (2.13), if we add the first equation with the second and subtract from the third equation, in both members, we will obtain (considering the fact that a^i_{jk} are symmetric)

$$a^i_{jk} = \frac{1}{2}(\Gamma^j_{ki} + \Gamma^k_{ji} - \Gamma^k_{ij} - \Gamma^i_{kj} - \Gamma^i_{jk} - \Gamma^j_{ik}).$$

It follows that

$$\omega^i_j = \frac{1}{2}\sum_{k=1}^{n}(\Gamma^i_{jk} + \Gamma^j_{ki} + \Gamma^k_{ji} - \Gamma^i_{kj} - \Gamma^j_{ik} - \Gamma^k_{ij})\theta^k,$$

and note that $\omega^j_i = -\omega^i_j$. This demonstrates the existence and uniqueness of connection 1-forms ω^i_j. □

Corollary **2.1.** *Let M and $\tilde{M}$ be two Riemannian manifolds of dimension n and let $f : M \longrightarrow \tilde{M}$ be an isometry. Let $(\tilde{\theta}^1, \ldots, \tilde{\theta}^n)$ be an adapted coframe in $\tilde{M}$ whose connection forms are $\tilde{\omega}^i_j$. If ω^i_j are the connection forms of M in the adapted coframe $(\theta^1, \ldots, \theta^n)$ where $\theta^i = f^*\tilde{\theta}^i$, for $i = 1, \ldots, n$, then $\omega^i_j = f^*\tilde{\omega}^i_j$.*

Proof. According to Proposition 2.1, in $\tilde{M}$, the connection 1-forms $\tilde{\omega}^i_j$ are the only one satisfying the structural equations

$$\mathrm{d}\tilde{\theta}^i + \sum_{j=1}^{n} \tilde{\omega}^i_j \wedge \tilde{\theta}^i = 0. \tag{2.14}$$

Again, by Proposition 2.1 applied to M, the connection 1-forms ω^i_j are the only one satisfying the structural equations

$$\mathrm{d}\theta^i + \sum_{j=1}^{n} \omega^i_j \wedge \theta^i = 0, \tag{2.15}$$

where $\theta^i = f^*\tilde{\theta}^i$, for $i = 1, \ldots, n$.

Calculating the pullback f^* in equation (2.14) and comparing with (2.15), by uniqueness of connection forms, we conclude that

$$\omega^i_j = f^*\tilde{\omega}^i_j,$$

as we wished. □

Theorem **2.1.** *Let S and $\tilde{S}$ be two surfaces in $\mathbb{S}^3$ and let $f : \mathbb{S}^3 \longrightarrow \mathbb{S}^3$ be an isometry such that $f(S) \subset \tilde{S}$. In these conditions, the following assertions are true:*

(i) *If K and $\tilde{K}$ are the Gaussian curvatures of S and $\tilde{S}$, respectively, then $K(p) = \tilde{K}\big(f(p)\big)$, for all $p \in S$.*

(ii) *If* λ_1, λ_2 *and* $\tilde{\lambda}_1, \tilde{\lambda}_2$ *are the principal curvatures of* S *and* $\tilde{S}$, *respectively, then* $\lambda_1(p) = \tilde{\lambda}_1\big(f(p)\big)$ *and* $\lambda_2(p) = \tilde{\lambda}_2\big(f(p)\big)$, *for all* $p \in S$.

Proof. Let us prove (i). Let $(\tilde{\theta}^1, \tilde{\theta}^2, \tilde{\theta}^3)$ be an adapted orthonormal coframe to the surface $\tilde{S}$. From the first structural equations, we detach the following

$$\begin{aligned} \mathrm{d}\tilde{\theta}^1 + \tilde{\omega}^1_2 \wedge \tilde{\theta}^2 = 0, \\ \mathrm{d}\tilde{\theta}^2 + \tilde{\omega}^2_1 \wedge \tilde{\theta}^1 = 0, \end{aligned} \tag{2.16}$$

and

$$\mathrm{d}\tilde{\omega}^1_2 = \tilde{K}\,\tilde{\theta}^1 \wedge \tilde{\theta}^2. \tag{2.17}$$

Since f is an isometry, the ternary $(\theta^1, \theta^2, \theta^3)$, where $\theta^i = f^*(\tilde{\theta}^i)$, for $i = 1, 2, 3$, defines an adapted orthonormal coframe to the surface S. Again, for this coframe, we detach the following structural equations

$$\begin{aligned} \mathrm{d}\theta^1 + \omega^1_2 \wedge \theta^2 = 0, \\ \mathrm{d}\theta^2 + \omega^2_1 \wedge \theta^1 = 0, \end{aligned} \tag{2.18}$$

and

$$\mathrm{d}\omega^1_2 = K\,\theta^1 \wedge \theta^2. \tag{2.19}$$

Applying the pullback f^* in equations (2.16) and (2.17), we obtain

$$\begin{aligned} \mathrm{d}\theta^1 + f^*(\tilde{\omega}^1_2) \wedge \theta^2 = 0, \\ \mathrm{d}\theta^2 + f^*(\tilde{\omega}^2_1) \wedge \theta^1 = 0, \end{aligned} \tag{2.20}$$

and

$$\mathrm{d}\big(f^*(\tilde{\omega}^1_2)\big) = \tilde{K}(f)\,\theta^1 \wedge \theta^2. \tag{2.21}$$

By Proposition 2.1 and according to the first structural equations in (2.18) and (2.20), we conclude that $\omega^1_2 = f^*(\tilde{\omega}^1_2)$, which together with the Gauss equations (2.19) and (2.21), generates $K = \tilde{K}(f)$, i.e., $K(p) = \tilde{K}\big(f(p)\big)$, for all $p \in S$.

Finally, to prove item (ii), let $(\tilde{\theta}^1, \tilde{\theta}^2, \tilde{\theta}^3)$ be an adapted orthonormal coframe to the surface $\tilde{S}$. In an analogous way, the set $\theta^i = f^*(\tilde{\theta}^i)$ for $i = 1, 2, 3$ is an adapted orthonormal coframe to the surface S. We have the following expressions

$$\mathrm{d}\tilde{\theta}^3 + \tilde{\omega}^3_1 \wedge \tilde{\theta}^1 + \tilde{\omega}^3_2 \wedge \tilde{\theta}^2 = 0, \tag{2.22}$$

and

$$\mathrm{d}\theta^3 + \omega^3_1 \wedge \theta^1 + \omega^3_2 \wedge \theta^2 = 0. \tag{2.23}$$

By Corollary 2.1, and applying the pullback f^* in equation (2.22), we obtain, by direct comparison with equation (2.23),

$$\omega_1^3 = f^*(\tilde{\omega}_1^3) \quad \text{and} \quad \omega_2^3 = f^*(\tilde{\omega}_2^3). \tag{2.24}$$

Now, writing

$$\omega_1^3 = \lambda_1 \theta^1 \quad \text{and} \quad \omega_2^3 = \lambda_2 \theta^2, \tag{2.25}$$

while we also have

$$\tilde{\omega}_1^3 = \tilde{\lambda}_1 \tilde{\theta}^1 \quad \text{and} \quad \tilde{\omega}_2^3 = \tilde{\lambda}_2 \tilde{\theta}^2. \tag{2.26}$$

Applying f^* in (2.26) and using (2.24), we conclude

$$\omega_1^3 = \tilde{\lambda}_1(f)\, \theta^1 \quad \text{and} \quad \omega_2^3 = \tilde{\lambda}_2(f)\, \theta^2, \tag{2.27}$$

and therefore, comparing the last equations with expressions in (2.25), we determine that

$$\lambda_1 = \tilde{\lambda}_1(f) \quad \text{and} \quad \lambda_2 = \tilde{\lambda}_2(f),$$

this immediately implies that $\lambda_1(p) = \tilde{\lambda}_1\big(f(p)\big)$ and $\lambda_2(p) = \tilde{\lambda}_2\big(f(p)\big)$, for all $p \in S$. □

Corollary **2.2.** *If S is an extrinsic homogeneous surface of $\mathbb{S}^3$, then we have*

(i) *its Gaussian curvature K is a constant.*
(ii) *its principal curvatures λ_1 and λ_2 are constant functions.*

Proof. Item (i). Since S is a homogeneous surface, then for any $p, q \in S$ there exists an isometry $f : \mathbb{S}^3 \longrightarrow \mathbb{S}^3$ such that $f(p) = q$. Now, by the last Theorem, putting $\tilde{S} = S$ (and hence $\tilde{K} = K$), we have $K(p) = K\big(f(p)\big)$, for all $p \in S$. Therefore, it results that $K(p) = K(q)$, for all $p, q \in S$ and thus $K(p)$ is constant on S.

Item (ii). In a similar way of the item (i), letting $\tilde{S} = S$ and taking $p, q \in S$, there exists $f \in \mathrm{Isom}(\mathbb{S}^3)$ such that $q = f(p)$. Thus we will have $\lambda_1(p) = \lambda_1\big(f(p)\big) = \lambda_1(q)$ and $\lambda_2(p) = \lambda_2\big(f(p)\big) = \lambda_2(q)$, therefore λ_1 and λ_2 are constants on S. □

3. Classification of Homogeneous Surfaces of $\mathbb{S}^3$

Proposition **3.1.** *If S is an umbilic surface, then its principal curvatures are constants. In particular, K will be constant.*

Proof. Since S is umbilic, i.e., $\lambda_1 = \lambda_2 = \lambda$, the equations in (2.6) reduce to $\mathrm{d}\lambda \wedge \theta^j = 0$, for $j = 1, 2$. Thus $\mathrm{d}\lambda = 0$, and hence λ is constant. Moreover, since $K = 1 + \lambda^2$, it results that K is also a constant. □

Proposition **3.2.** *If the principal curvatures of S are constants, then or S is umbilic or S has null Gaussian curvature K.*

Proof. Suppose that λ_1 and λ_2 are constants. From the equations in (2.6) it follows that $(\lambda_1 - \lambda_2)\omega_1^2 \wedge \theta^j = 0$, for $j = 1, 2$. Thus, there are two possibilities: or $\lambda_1 = \lambda_2$, and hence S is umbilic, or $\omega_1^2 \wedge \theta^j = 0$, for $j = 1, 2$, which implies $\omega_1^2 = 0$, and hence $\mathrm{d}\omega_1^2 = 0$. However, looking to the equation (2.2) we conclude that $K\theta^1 \wedge \theta^2 = 0$. Therefore $K = 0$, and hence S is a surface with null Gaussian curvature. $\square$

3.1. **The case** $\lambda_1 = \lambda_2$**.** Let $S \subset \mathbb{S}^3$ be a homogeneous surface, and hence, by Corollary 2.2, its principal curvatures $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$. Let (e_1, e_2) be the principal direction of S and e_3 the normal field of S. Remember that in these conditions, $\theta^3 \equiv 0$ over S. Since the frame (e_1, e_2, e_3) is adapted to S, we have the following set of equations

$$\begin{aligned}
\mathrm{d}e_1 &= \omega_1^2 e_2 - \lambda\theta^1 e_3 + \theta^1 e_4,\\
\mathrm{d}e_2 &= \omega_2^1 e_1 - \lambda\theta^2 e_3 + \theta^2 e_4,\\
\mathrm{d}e_3 &= \lambda\theta^1 e_1 + \lambda\theta^2 e_2 = \lambda\,\mathrm{id},\\
\mathrm{d}e_4 &= -\theta^1 e_1 - \theta^2 e_2 = -\mathrm{id},
\end{aligned}$$

moreover, note that $\omega_3^1 = \lambda\theta^1$ and $\omega_3^2 = \lambda\theta^2$.

Then, consider the following vector field defined on S

$$X = \mathbf{x} - \frac{1}{\lambda}e_3,$$

where $\mathbf{x}$ is a parametrization of S. We will show that $X = \mathbf{x}_0 =$ constant on S. In fact, differentiating X we obtain

$$\begin{aligned}
\mathrm{d}X &= \mathrm{d}\Big(\mathbf{x} - \frac{1}{\lambda}e_3\Big)\\
&= \theta^1 e_1 + \theta^2 e_2 - \frac{1}{\lambda}(\lambda\theta^1 e_1 + \lambda\theta^2 e_2)\\
&= \theta^1 e_1 + \theta^2 e_2 - \theta^1 e_1 - \theta^2 e_2 = 0.
\end{aligned}$$

Thus, $\mathrm{d}X = 0$ on S, and hence $X = \mathbf{x}_0 =$ constant. If we write

$$\mathbf{x}_0 = \mathbf{x} - \frac{1}{\lambda}e_3 \quad \text{which implies} \quad \mathbf{x} - \mathbf{x}_0 = \frac{1}{\lambda}e_3,$$

Taking the norm, in both members, on the last equation, we conclude

$$||\mathbf{x} - \mathbf{x}_0|| = \frac{1}{\lambda}, \tag{3.1}$$

which immediately implies that

$$\langle \mathbf{x} - \mathbf{x}_0, \mathbf{x} - \mathbf{x}_0 \rangle = \frac{1}{\lambda^2}, \tag{3.2}$$

and this is an equation of a sphere with center $\mathbf{x}_0$ and radius $\frac{1}{|\lambda|}$.

Remember that S is a connected homogeneous surface, therefore it is a *complete* surface. Hence S is a whole 2-sphere.

3.2. **The case $\lambda_1 \neq \lambda_2$.** Let $S \subset \mathbb{S}^3$ be a surface, and suppose its principal curvatures λ_1, λ_2 are constant and distinct. From equations of Mainardi-Codazzi (2.6), we have

$$(\lambda_1 - \lambda_2)\omega_1^2 \wedge \theta^2 = 0,$$
$$(\lambda_1 - \lambda_2)\omega_1^2 \wedge \theta^1 = 0.$$

Therefore, it follows that $\omega_1^2 = 0$. From equation of Gauss (2.2), follows that $K\theta^1 \wedge \theta^2 = 0$, which implies that $K = 0$, but since $K = 1 + \lambda_1\lambda_2$, it results the condition:

$$\lambda_1\lambda_2 = -1.$$

Since $K = 0$, there exists a parametrization $\mathbf{x} : U \ni (u^1, u^2) \longmapsto \mathbb{R}^4$, where U is an open set of $\mathbb{R}^2$, $\mathbf{x}(U) \subset S$, and satisfying the property

$$\frac{\partial \mathbf{x}}{\partial u^1} = e_1 \quad \text{and} \quad \frac{\partial \mathbf{x}}{\partial u^2} = e_2.$$

In these conditions, we know that

$$\mathrm{d}u^1 = \theta^1 \quad \text{and} \quad \mathrm{d}u^2 = \theta^2, \tag{3.3}$$

since $(\theta^1, \theta^2, \theta^3, \theta^4)$ is the dual base of (e_1, e_2, e_3, e_4).

On the other hand, we have the following system of equations

$$\begin{aligned}
\mathrm{d}e_1 &= \lambda_1\theta^1 e_3 + \theta^1 e_4 = (\lambda_1 e_3 + e_4)\mathrm{d}u^1,\\
\mathrm{d}e_2 &= \lambda_2\theta^2 e_3 + \theta^2 e_4 = (\lambda_2 e_3 + e_4)\mathrm{d}u^2,\\
\mathrm{d}e_3 &= -\lambda_1\theta^1 e_1 - \lambda_2\theta^2 e_2 = -\lambda_1 e_1\,\mathrm{d}u^1 - \lambda_2 e_2\,\mathrm{d}u^2,\\
\mathrm{d}e_4 &= -\theta^1 e_1 - \theta^2 e_2 = -e_1\,\mathrm{d}u^1 - e_2\,\mathrm{d}u^2.
\end{aligned}$$

Then, let us consider the following vector fields

$$\begin{aligned}
f_1 &= e_1,\\
f_2 &= e_2,\\
f_3 &= \frac{\lambda_1 e_3 + e_4}{\sqrt{\lambda_1^2 + 1}},\\
f_4 &= \frac{\lambda_2 e_3 + e_4}{\sqrt{\lambda_2^2 + 1}},
\end{aligned}$$

it is easy to check that $\langle f_i, f_j\rangle = \delta_{ij}$, i.e., the set (f_1, f_2, f_3, f_4) forms a base of $\mathbb{R}^4$.

Differentiating the vector fields f_i, we obtain

$$\begin{aligned}
\mathrm{d}f_1 &= \mathrm{d}e_1 = \sqrt{\lambda_1^2+1}\,\theta^1 f_3 = \left(\sqrt{\lambda_1^2+1}\,\mathrm{d}u^1\right) f_3,\\
\mathrm{d}f_2 &= \mathrm{d}e_2 = \sqrt{\lambda_2^2+1}\,\theta^2 f_4 = \left(\sqrt{\lambda_2^2+1}\,\mathrm{d}u^2\right) f_4,\\
\mathrm{d}f_3 &= \mathrm{d}\left(\frac{\lambda_1 e_3 + e_4}{\sqrt{\lambda_1^2+1}}\right) = -\sqrt{\lambda_1^2+1}\,\theta^1 f_1 = -\left(\sqrt{\lambda_1^2+1}\,\mathrm{d}u^1\right) f_1,\\
\mathrm{d}f_4 &= \mathrm{d}\left(\frac{\lambda_2 e_3 + e_4}{\sqrt{\lambda_2^2+1}}\right) = -\sqrt{\lambda_2^2+1}\,\theta^2 f_2 = -\left(\sqrt{\lambda_2^2+1}\,\mathrm{d}u^2\right) f_2,
\end{aligned}$$

and we will denote by $k_i = \sqrt{\lambda_i^2+1}$, for $i = 1, 2$, just for simplify the expressions above. So, we rewrite

$$\begin{aligned}
\mathrm{d}f_1 &= k_1\,\mathrm{d}u^1\, f_3,\\
\mathrm{d}f_2 &= k_2\,\mathrm{d}u^2\, f_4,\\
\mathrm{d}f_3 &= -k_1\,\mathrm{d}u^1\, f_1,\\
\mathrm{d}f_4 &= -k_2\,\mathrm{d}u^2\, f_2.
\end{aligned} \tag{3.4}$$

Thus, if we observe the equations in (3.4) and the fact that $e_i = f_i$ for $i = 1, 2$, we conclude that

$$\begin{cases} \frac{\partial f_1}{\partial u^1} &= k_1 f_3,\\ \frac{\partial f_1}{\partial u^2} &= 0, \end{cases} \qquad \begin{cases} \frac{\partial f_2}{\partial u^1} &= 0,\\ \frac{\partial f_2}{\partial u^2} &= k_2 f_4, \end{cases}$$

$$\begin{cases} \frac{\partial f_3}{\partial u^1} &= -k_1 f_1,\\ \frac{\partial f_3}{\partial u^2} &= 0, \end{cases} \qquad \begin{cases} \frac{\partial f_4}{\partial u^1} &= 0.\\ \frac{\partial f_4}{\partial u^2} &= -k_2 f_2, \end{cases}$$

Consider the following curve on S:

$$c_1(u^1) = \mathbf{x}(u^1, u_0^2),$$

where u_0^2 is fixed. Then

$$X = c_1(u^1) + \frac{1}{k_1} f_3,$$

is a vector field defined on S. We have now

$$\begin{aligned}
\frac{\partial X}{\partial u^1} &= \frac{\partial}{\partial u^1}\left(c_1(u^1) + \frac{1}{k_1} f_3\right)\\
&= \frac{\partial \mathbf{x}}{\partial u^1}(u^1, u_0^2) + \frac{1}{k_1}\frac{\partial f_3}{\partial u^1}\\
&= f_1 + \frac{1}{k_1}(-k_1 f_1) = 0.
\end{aligned}$$

Therefore X is a constant vector field, i.e., we can write

$$c_1(u^1) + \frac{1}{k_1} f_3 = p_0,$$

where p_0 is a point of S.

It follows that

$$c_1(u^1) + \frac{1}{k_1} f_3 = p_0 \text{ which implies } c_1(u^1) - p_0 = -\frac{1}{k_1} f_3;$$

and taking the norm, in both members, we obtain

$$\left\| c_1(u^1) - p_0 \right\| = \frac{1}{k_1} ||f_3|| \text{ which implies } \left\| c_1(u^1) - p_0 \right\| = \frac{1}{k_1},$$

and this is an equation of a circle of center p_0 and radius $\frac{1}{k_1}$.

We will show that this circle belongs to a special plane. In fact, we have

$$\begin{aligned} \frac{\partial}{\partial u^1} \langle c_1(u^1), f_2 \rangle &= \left\langle \frac{\partial \mathbf{x}}{\partial u^1}, f_2 \right\rangle + \left\langle \mathbf{x}(u^1, u_0^2), \frac{\partial f_2}{\partial u^1} \right\rangle \\ &= \langle f_1, f_2 \rangle + \left\langle c_1(u^1), 0 \right\rangle = 0, \end{aligned}$$

and hence $\left\langle c_1(u^1), f_2 \right\rangle = a$, where $a \in \mathbb{R}$.

On the other hand,

$$\begin{aligned} \frac{\partial}{\partial u^1} \langle c_1(u^1), f_4 \rangle &= \left\langle \frac{\partial \mathbf{x}}{\partial u^1}, f_4 \right\rangle + \left\langle \mathbf{x}(u^1, u_0^2), \frac{\partial f_4}{\partial u^1} \right\rangle \\ &= \langle f_1, f_4 \rangle + \left\langle c_1(u^1), 0 \right\rangle = 0, \end{aligned}$$

and hence $\left\langle c_1(u^1), f_4 \right\rangle = b$, where $b \in \mathbb{R}$.

The last pair of equations defines a plane π_1 in $\mathbb{R}^4$, i.e.,

$$\pi_1 : \quad \left\langle c_1(u^1), f_2 \right\rangle = a, \quad \left\langle c_1(u^1), f_4 \right\rangle = b, \qquad a, b \in \mathbb{R}.$$

Therefore, it results that the circle

$$\mathcal{C}_1 : \quad \left\langle c_1(u^1) - p_0, c_1(u^1) - p_0 \right\rangle = \frac{1}{k_1^2},$$

belongs to plane π_1.

In a similar way, consider the following curve in S:

$$c_2(u^2) = \mathbf{x}(u_0^1, u^2),$$

where u_0^1 is fixed. Let

$$Y = c_2(u^2) + \frac{1}{k_2} f_4,$$

be a vector field on S. We have

$$\begin{aligned}\frac{\partial Y}{\partial u^2} &= \frac{\partial}{\partial u^2}\left(c_2(u^2) + \frac{1}{k_2}f_4\right)\\ &= \frac{\partial \mathbf{x}}{\partial u^2}(u_0^1, u^2) + \frac{1}{k_2}\frac{\partial f_4}{\partial u^2}\\ &= f_2 + \frac{1}{k_2}(-k_2 f_2) = 0.\end{aligned}$$

Therefore Y is a constant vector field, i.e., we can write

$$c_2(u^2) + \frac{1}{k_2}f_4 = q_0,$$

where q_0 is a point of S.

It follows that,

$$c_2(u^2) + \frac{1}{k_2}f_4 = p_0 \text{ which implies } c_2(u^2) - q_0 = -\frac{1}{k_2}f_4;$$

and again, taking the norm, in both members, we obtain

$$\big\|c_2(u^2) - q_0\big\| = \frac{1}{k_2}||f_4|| \text{ which implies } \big\|c_2(u^2) - q_0\big\| = \frac{1}{k_2},$$

and this is an equation of a circle centered in q_0 and with radius $\frac{1}{k_2}$.

Again, we will show that this circle belongs to a special plane. In fact, we have

$$\begin{aligned}\frac{\partial}{\partial u^2}\langle c_2(u^2), f_1\rangle &= \Big\langle \frac{\partial \mathbf{x}}{\partial u^2}, f_1\Big\rangle + \Big\langle \mathbf{x}(u_0^1, u^2), \frac{\partial f_1}{\partial u^2}\Big\rangle\\ &= \langle f_2, f_1\rangle + \big\langle c_2(u^2), 0\big\rangle = 0,\end{aligned}$$

and hence $\big\langle c_2(u^2), f_1\big\rangle = c$, where $c \in \mathbb{R}$.

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial u^2}\langle c_2(u^2), f_3\rangle &= \Big\langle \frac{\partial \mathbf{x}}{\partial u^2}, f_3\Big\rangle + \Big\langle \mathbf{x}(u_0^1, u^2), \frac{\partial f_3}{\partial u^2}\Big\rangle\\ &= \langle f_2, f_1\rangle + \big\langle c_2(u^2), 0\big\rangle = 0,\end{aligned}$$

and hence $\big\langle c_2(u^2), f_3\big\rangle = d$, where $d \in \mathbb{R}$.

The last pair of equations defines a plane π_2 in $\mathbb{R}^4$, i.e.,

$$\pi_2 : \quad \big\langle c_2(u^2), f_1\big\rangle = c, \quad \big\langle c_2(u^2), f_3\big\rangle = d, \qquad c, d \in \mathbb{R}.$$

Finally, it results that the circle

$$\mathcal{C}_2 : \quad \big\langle c_2(u^2) - q_0, c_2(u^2) - q_0\big\rangle = \frac{1}{k_2^2},$$

belongs to the plane π_2.

Since $\{f_1, f_3\}$ and $\{f_2, f_4\}$ belong to mutually orthogonal planes, the circles $\mathcal{C}_1$ and $\mathcal{C}_2$ are both orthogonal, and hence they generate a torus in $\mathbb{S}^3$.

Another time, keeping in mind that S is a connected homogeneous surface, it follows that S is a *complete* surface, and hence S is a whole torus.

4. Conclusions

From Corollary 2.2 and from both cases (i), (ii), we conclude that complete immersed surfaces with constant principal curvatures are 2-spheres and torus. Since these surfaces are homogeneous, we have the following classification theorem.

Theorem **4.1.** *If S is a regular connected immersed homogeneous surface of $\mathbb{S}^3$, then S is one and only one surface between the following types:*

(i) *S is an immersed 2-sphere in $\mathbb{S}^3$.*
(ii) *S is an immersed flat torus ($\cong S^1 \times S^1$) in $\mathbb{S}^3$.*

References

[1] **do Carmo, M. P.**, *Differential Forms and Applications*, Springer-Verlag (UTM).
[2] **Guggenheimer, H. W.**, *Differential Geometry*, Dover Pub., New York, 1963.
[3] **Hopf, H.**, *Differential Geometry in the Large*, Lecture Notes in Mathematics (Vol 1000).
[4] **O'Neil, B.**, *Elementary Differential Geometry*, Academic Press, New York, 1966.
[5] **Verderesi, J. A.**, *Notas Manuscritas.*

Departamento de Matemática,
Universidade de São Paulo, Brazil
E-mail address: maccori@ime.usp.br

Departamento de Matemática,
Universidade de São Paulo, Brazil
E-mail address: javerd@ime.usp.br